Generalized step iteration in the relaxation method for the feasibility problem.*

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Abstract

The problem of finding a feasible solution to a linear inequality system arises in numerous contexts. In González-Gutiérrez and Todorov [6], an algorithm, called extended relaxation method, for solving the feasibility problem has been proposed by the authors. Convergence of the algorithm has been proven. Later on, we have proved [8] linear convergence of a class of extended relaxation methods depending on a parameter. In this paper, we shrink this class of extended relaxation methods to just one, generalizing the step iteration. We prove convergence and investigate the rate of convergence. Numerical experiments have been provided, as well.

1 Introduction

We deal with *linear semi-infinite systems* (LSIS's for short) of the form:

$$\sigma = \{a_t' x \geq b_t, t \in T\} \tag{1}$$

where, T is an arbitrary nonempty index set, $x \in \mathbb{R}^n$, $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are arbitrary mappings. We denote by F the feasible set of σ.

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One of the central problems of linear semi-infinite systems is to find a solution of this type of systems (called *feasibility problem*) for the case of ordinary systems (finite number of constraints and variables), this problem is equivalent to the Linear Programming Problem, however, this equivalence disappears when the number of constraints or variables is infinite, so, the numerical methods solving LSIS are, to some extent, independent of the methods for problems of semi-infinite linear programming ([4]).

Numerical methods for some kind of LSIS in $\mathbb{R}^{(T)}$ (the space of generalized sequences) arising in economic theory, which are called linear systems differences, can be found in [11]. In [5] the extended elimination method of Fourier is studied, this, from an external representation of some closed set (an external representation of the projection of the feasible set by means of a system), this can be F, in a certain sense, a smaller system is obtained. The extended Fourier method is important in the theory of linear semi-infinite systems (used in some proofs [5]). Moreover, about of the theoretic study of LSIS, [5], the well-known representation Theorem of Motzkin is extended to LSIS when the feasible set is written as a sum of a polytope (convex hull of a finite set) and of a finitely generated convex cone, which is an internal representation of F. The constructive proof of this result (representation Theorem of Motzkin) can be seen as a method solving LSIS.

In the present paper, for a given LSIS's, σ, , we assume various initial conditions. At the beginning, that $a_t \neq 0_n$ for all $t \in T$, so, each inequality represents a closed half-space. Furthermore, we suppose that F is not empty.

Considering $x^k \notin F$, for some $k \in \mathbb{N}$, the usual method for solving the feasibility problem, is an iterative process, where the next iteration is found as follows:

$$x^{k+1} = x^k + \lambda\mu\frac{a}{\|a\|},$$

where μ is the distance from x^k fo the set F, $\lambda \in \mathbb{R}$ is a given parameter of relaxation and a is a certain vector in $\mathbb{R}^n$. The projection algorithm reaches its goal with any special choice of the whole family of parameters, by performing projections onto different sets on each iteration step. In recent papers (see [8], [6], [9] and [7]) the authors have investigated this class of methods of projections which work with a fixed value of $\lambda \in (0, 2]$, as a relaxation parameter during the whole process and numerical reports about the performance of the whole class of algorithms depending on the fixed parameter λ in each iteration were given.

In this work we deal with a more general relaxation process, that those appearing in [7] and [8], in the sense that the parameter λ appearing in the Algorithm 2 is chosen as one parametric function $\lambda : \mathbb{R} \to [\nu, 2]$, with $\nu \in (0, 2)$ where $\lambda = \lambda(r)$, we denote $\lambda_r = \lambda(r)$, for short, in each iteration $r = 0, 1, 2, \ldots$. On the basis of this scheme, first, we prove the convergence of the algorithm and then, we show that the algorithm has linear convergence.

The combination of the different values of parameters in each iterations, significantly increase the algorithm performance and better numerical results that those which appeared in [7] are obtained.

The paper is organized as follows, in Section 2, we present the procedure for finding a feasible solution of σ, and we prove the convergence under the assumptions established above. In Section 3, we present the theorems that guaranteeing the linear convergence of the method. Section 4, presents a numerical treatment with some examples.

2 Projection algorithm and analysis of convergence

In this section we present the general relaxation algorithm depending on a different λ_r, for each step $r = 0, 1, 2, \ldots$ and we establish the main result of this work.

Algorithm 1 *Modified extended relaxation algorithm for the feasibility problem:*

1. *Choose the parameters* $M > 2$ *and* $\beta > 0$; *choose an arbitrary vector* $x^0 \in \mathbb{R}^n$. *Set the iteration index* $r = 0$, *and choose* $\lambda_r \in [\nu, 2]$ *with* $\nu \in (0, 2)$.

2. *Minimize the slack function* $g(t, x) = a_t' x - b_t$ *at* x^r, *finding* $u_r = \inf_{t \in T} g(t, x^r)$. *If* $u_r \geq 0$, *stop* ($x^r \in F$). *Otherwise, take the index set* $T_r = \{t \in T | g(t, x^r) < 0\}$ *(indexes of violated inequalities by* x^r*).*

3. *Set* $\beta_r = \beta$ *and consider the global optimization problem*

$$\sup \left\{ \frac{b_t - a_t' x^r}{\|a_t\|}, t \in T_r \right\} = \mu_r. \qquad (2)$$

4. *Furthermore, find one β_r approximation, ε_r, of the solution, μ_r, of the problem (2) $(\mu_r - \beta_r < \varepsilon_r \leq \mu_r)$. If $\beta_r < \varepsilon_r(M-1)$, then*

$$\frac{\mu_r}{M} < \varepsilon_r := \frac{b_{t_r} - a'_{t_r}x^r}{\|a_{t_r}\|} \leq \mu_r, \text{ for some } t_r \in T_r,$$

and choose $x^{r+1} = x^r + \lambda_r\varepsilon_r\frac{a_{t_r}}{\|a_{t_r}\|}$. Replace r by $r+1$ and loop to step 2. If not, set $\beta_r = \beta_r/2$ and go to the step 4.

Remark 2 *In [7], it has been shown that in the algorithm ε_r always exists for all $r \in \mathbb{N}$ and it takes finite values different from zero.*

The proof of the convergence for the case $\lambda_r = \lambda$ fixed for each $r = 0, 1, 2, \ldots$ has been given in [7]. In the present work, we prove the convergence of the algorithm for the more general case, when $\lambda_r \in [\nu, 2]$ with a fixed $\nu \in (0, 2)$ for $r = 0, 1, \ldots$.

The following result has been proven in a similar way, as it was done in Theorem 3, presented in [7].

Theorem 3 *Let σ be a consistent system such that $\dim F = n$. Given an initial point, $x^0 \in \mathbb{R}^n$ and a fixed real number ν, such that $\nu \in (0, 2)$. If for each $r = 0, 1, 2, \ldots$ we chose an arbitrary $\lambda_r \in [\nu, 2]$, the Algorithm 1 either ends after a finite number of steps, or generates an infinite sequence, $\{x^r\}$, converging to some element of F.*

Proof. Our proof starts with the observation that if the sequence is finite and the last point belongs to F then we finish. It remains to prove that the convergence holds if we assume that $\{x^r\}$ is an infinite sequence of infeasible points.

For each $t \in T$ we denote $H_t = \{x \in \mathbb{R}^n \mid a'_t x = b_t\}$. We have $\mu_r > 0$, for all $r \in \mathbb{N}$, i.e., $x^r \notin H_{t_r}$, then the point x^{r+1} is along the vector a_{t_r} starting from x^r. The distance between the two points is $\lambda_r\varepsilon_r$.

By hypothesis, there exist $z \in \mathbb{R}^n$ and $\delta > 0$ such that the open ball $B_\delta(z)$ of center z and radius δ satisfies

$$B_\delta(z) \subset F \subset \{x \in \mathbb{R}^n \mid a'_{t_r}x \geq b_{t_r}\}, \quad r = 1, 2, \ldots$$

and $\rho_{t_r}:=d(z, H_{t_r}) \geq \delta$.

By construction, the line determined by x^r and x^{r+1} is orthogonal to H_{t_r}. Let h_r be the distance from z to that line. Consider the affine hull of

$\{x^r, x^{r+1}, z\}$. We elect a coordinate system in this hyperplane, with abscises axis, the line throughout the points x^r and x^{r+1}, directed in that way, and ordinates axis, the perpendicular to the line throughout the points x^r and x^{r+1}, directed in such a way that z belongs to the first orthant. With this oriented system, the coordinates of the points x^r , x^{r+1} and z are $(-\varepsilon_r, 0)$, $((\lambda_r - 1)\varepsilon_r, 0) = (\xi_r\varepsilon_r, 0)$, where $\lambda_r - 1 = \xi_r \in (-1, 1]$, and (ρ_{t_r}, h_r), respectively, with $h_r \geq 0$ (the case when the dimension of the affine hull is 1 and $h_r = 0$ is trivial). Then

$$\begin{aligned}\|x^r - z\|^2 - \|x^{r+1} - z\|^2 &= \left[(\rho_{t_r} + \varepsilon_r)^2 + h_r^2\right] - \left[(\rho_{t_r} - \xi_r\varepsilon_r)^2 + h_r^2\right] \\ &= (1 - \xi_r^2)\varepsilon_r^2 + 2(1 + \xi_r)\rho_{t_r}\varepsilon_r.\end{aligned}$$

Hence, for $r \in \mathbb{N}$, we have

$$0 \leq \|x^{r+1} - z\|^2 = \|x^r - z\|^2 - (1 - \xi_r^2)\varepsilon_r^2 - 2(1 + \xi_r)\rho_{t_r}\varepsilon_r.$$

Since $-\rho_{t_r} \leq -\delta$ we obtain

$$0 \leq \|x^{r+1} - z\|^2 \leq \|x^r - z\|^2 - (1 - \xi_r^2)\varepsilon_r^2 - 2(1 + \xi_r)\delta\varepsilon_r.$$

So, we can consider the above inequalities for the first $r - 1$ terms, i.e., for each $k = 0, \ldots, r - 1$

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - (1 - \xi_k^2)\varepsilon_k^2 - 2(1 + \xi_k)\delta\varepsilon_k,$$

written in a different way,

$$\|x^{k+1} - z\|^2 - \|x^k - z\|^2 \leq -(1 - \xi_k^2)\varepsilon_k^2 - 2(1 + \xi_k)\delta\varepsilon_k.$$

Adding these inequalities for $k = 0, \ldots, r - 1$ we get

$$\sum_{k=1}^{r-1} \left(\|x^{k+1} - z\|^2 - \|x^k - z\|^2\right) \leq -\sum_{k=1}^{r-1}(1 - \xi_k^2)\varepsilon_k^2 - 2\delta\sum_{k=1}^{r-1}(1 + \xi_k)\varepsilon_k. \quad (3)$$

In the right hand we have a telescopic series

$$\sum_{k=1}^{r-1} \left(\|x^{k+1} - z\|^2 - \|x^k - z\|^2\right) = \|x^r - z\|^2 - \|x^0 - z\|^2. \quad (4)$$

As a result, from (3) and (4),

$$\|x^r - z\|^2 - \|x^0 - z\|^2 \leq -\sum_{k=0}^{r-1}(1-\xi_k^2)\varepsilon_k^2 - 2\delta\sum_{k=0}^{r-1}(1+\xi_k)\varepsilon_k,$$

whereby,

$$2\delta\sum_{k=0}^{r-1}(1+\xi_k)\varepsilon_k \leq \sum_{k=0}^{r-1}(1-\xi_k^2)\varepsilon_r^2 + 2\delta\sum_{k=0}^{r-1}(1+\xi_k)\varepsilon_k \leq \|x^0 - z\|^2,$$

so that,

$$\sum_{k=0}^{r-1}(1+\xi_k)\varepsilon_k \leq \frac{1}{2\delta}\|x^0 - z\|^2.$$

Since $\xi_k = \lambda_k - 1$, by hypothesis $\nu \leq \lambda_k$ for a fixed value $\nu \in (0,2)$, then

$$\sum_{k=0}^{r-1}\nu\varepsilon_k \leq \sum_{k=0}^{r-1}\lambda_k\varepsilon_k \leq \frac{1}{2\delta}\|x^0 - z\|^2, \tag{5}$$

so

$$\sum_{k=0}^{r-1}\varepsilon_k \leq \frac{1}{2\delta\nu}\|x^0 - z\|^2,$$

If we consider the sequence $\eta_{r-1} = \sum_{k=0}^{r-1}\varepsilon_k$, and the constant $K = \frac{1}{2\nu\delta}\|x^0 - z\|^2$, we have $\eta_{r-1} \geq 0$ for all $r \in \mathbb{N}$, then $0 \leq \lim_r \eta_r \leq K$, i.e., the sequence $\{\eta_r\}$ is bounded and increasing, whereby it converges. Hence, $\sum_{r=0}^{\infty}\varepsilon_r$ converges as well (and $\lim_r \varepsilon_r = 0$).

Since, in Step 4 of the algorithm, we have chosen ε_r such that $0 < \frac{\mu_r}{M} < \varepsilon_r$, i.e., $0 < \mu_r < \varepsilon_r M$, we get $\lim_r \mu_r = 0$.

From (5) we have

$$\sum_{k=0}^{r-1}\lambda_k\varepsilon_k \leq \frac{1}{2\delta}\|x^0 - z\|^2,$$

but, in Step 4 of the algorithm, we have

$$\lambda_k\varepsilon_k = \|x^r - x^{r+1}\|,$$

so

$$\sum_{k=0}^{r-1}\|x^r - x^{r+1}\| \leq \frac{1}{2\delta}\|x^0 - z\|^2$$

and then the series $\sum_{r=0}^{\infty} \|x^r - x^{r+1}\|$ converges. Therefore, $\sum_{r=0}^{\infty}(x^r - x^{r+1})$ is absolutely convergent (see Th. 26.7 [2]), and, $\lim_r x^r = \hat{x}$, for some $\hat{x} \in \mathbb{R}^n$.

Finally, we will show that $\hat{x} \in F$. For any $t \in T$, and for all $r \in \mathbb{N}$ we have

$$\frac{b_t - a_t' x^r}{\|a_t'\|} \leq \begin{cases} \mu_r, & t \in T_r, \\ 0, & \text{otherwise.} \end{cases}$$

Taking limit in the above relation when $r \to \infty$, we get $\frac{b_t - a_t' \hat{x}}{\|a_t'\|} \leq 0$, for all $t \in T$, and this proves that $\hat{x} \in F$. ■

3 Rate of convergence of the algorithm

In this section, it is requiered that each iteration, $r \in \mathbb{N}$, $\lambda_r \in [\nu, 2]$ with $\nu \in (0, 2)$. So we show that Algorithm 1 has the rate of convergence linear. This fact is established in Theorem 7. Before to prove it, several statements along with some previous lemmas are presented.

Let us consider the sequence $\{x^r\}$ generated by the algorithm described in Section 2. Together with conditions in Theorem 3, we suppose some additional conditions on the nominal data a_t and b_t.

Let us denote by $B = \inf\{\|a_t\| : t \in T\} \geq 0$ and $N = \sup\{\|a_t\| : t \in T\} \leq \infty$, respectively.

Lemma 4 *In the definition of* $\mu_r = \sup\left\{\frac{b_t - a_t' x^r}{\|a_t\|} : t \in T_r\right\}$ *in (2) we can replace* T_r *with* T *for any* $r \in \mathbb{N}$.

The following result, Lemma 2.1 from [1], will be used in Theorem 7.

Lemma 5 *Let* $\lambda \in [0, 2]$ *and* x, y *be two points in* $\mathbb{R}^n$ *separated by the hyperplane* $H = \{x \in \mathbb{R}^n | a'x = b\}$, *such that* $a'x < b$ *and* $a'y \geq b$. *Then*

$$\left\|x + \lambda\left(\acute{x} - x\right) - y\right\|^2 \leq \|x - y\|^2 - \lambda\left(2 - \lambda\right)\left\|\acute{x} - x\right\|^2, \tag{6}$$

where $\acute{x}$ *is the orthogonal projection of* x *on* H. *The equality holds if* $\lambda = 0$, *or* $\lambda = 2$ *and* $y \in H$.

We need of a statement similar to Lemma 1 presented in [10].

Lemma 6 *If* $\operatorname{int} F \neq \emptyset$, $N < \infty$ *and* $B > 0$, *then there exists a constant* $0 < \gamma < 1$ *such that* $\mu_r \geq \gamma d(x^r, F)$ *for all* $r = 0, 1, 2, \ldots$.

Proof. We assume, temporarily, that $0_n \in \operatorname{int} F$. Therefore, there exists $\delta > 0$ such that $B_\delta(0) \subset F$, which implies that $d(0_n, H_t) \geq \delta$ for any $t \in T$, i.e., $-b_t/\|a_t\| \geq \delta$ for all $t \in T$. Hence, there exist $\alpha > 0$ satisfying

$$-b_t \geq \delta B = \alpha \quad \text{for all } t \in T. \tag{7}$$

For a fixed $r \in \mathbb{N}$; let be $x^r \in \{x^k\}$, and y^r be the point in F such that $\|x^r - y^r\| = d(x^r, F)$, i.e., y^r is the nearest point of F to x^r. It is well known that the inequality $(y^r - x^r)'z \geq (y^r - x^r)'y^r$ is a consequence of the system (1). By the Farkas's Lemma

$$\begin{pmatrix} y^r - x^r \\ (y^r - x^r)'y^r \end{pmatrix} \in \operatorname{cl}\operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T, \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Then, for a certain sequences $\{\beta_r^j\} \subset \mathbb{R}_+^{(T)}$ and $\{(\beta_r^j)_0\} \subset \mathbb{R}_+$, we can write

$$\begin{pmatrix} y^r - x^r \\ (y^r - x^r)'y^r \end{pmatrix} = \lim_j \left\{ \sum_{t \in T} \left(\beta_r^j\right)_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \left(\beta_r^j\right)_0 \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Thereby

$$y^r - x^r = \lim_j \sum_{t \in T} \left(\beta_r^j\right)_t a_t, \tag{8}$$

and

$$(y^r - x^r)'y^r = \lim_j \left(\sum_{t \in T} \left(\beta_r^j\right)_t b_t - \left(\lambda_r^j\right)_0 \right). \tag{9}$$

Therefore,

$$\lim_j \left(\sum_{t \in T} \left(\beta_r^j\right)_t a_t' y^r \right) - \lim_j \left(\sum_{t \in T} \left(\beta_r^j\right)_t b_t - \left(\beta_r^j\right)_0 \right) = 0.$$

As $(\beta_r^j)_t \geq 0$ and $a_t'y^r - b_t \geq 0$, for all $t \in T$,

$$\lim_j \left(\beta_r^j\right)_0 = 0, \tag{10}$$

and

$$\lim_j \left(\sum_{t \in T} \left(\beta_r^j\right)_t (a_t'y^r - b_t) \right) = 0. \tag{11}$$

We know that $x^r \to \bar{x}$ and $\|y^r - x^r\| \to 0$, therefore $L = \sup\{\|y^r\| : r = 0, 1, ...\} < +\infty$. From (7) we have

$$\sum_{t\in T} \left(\beta_r^j\right)_t \alpha \leq -\sum_{t\in T} \left(\beta_r^j\right)_t b_t, \text{, for all } j = 0, 1, 2, ...,$$

then $\alpha \liminf_j \sum_{t\in T} (\beta_r^j)_t \leq \liminf_j \left(-\sum_{t\in T} (\beta_r^j)_t b_t \right)$, so

$$\alpha \liminf_j \sum_{t\in T} \left(\beta_r^j\right)_t \leq -\limsup_j \sum_{t\in T} \left(\beta_r^j\right)_t b_t \tag{12}$$

and by (10) and (9) we have

$$-\limsup_j \sum_{t\in T} \left(\beta_r^j\right)_t b_t = (x^r - y^r)' y^r, \tag{13}$$

so, by (12) and (13) we have

$$\liminf_j \sum_{t\in T} \left(\beta_r^j\right)_t \leq \alpha^{-1} L \left\|x^r - y^r\right\|. \tag{14}$$

By (8), (11), (14) and Lemma 4 we have

$$\begin{aligned}
\|y^r - x^r\|^2 &= \lim_j \sum_{t\in T} \left(\beta_r^j\right)_t a_t'(y^r - x^r) \\
&= \lim_j \left(\sum_{t\in T} \left(\beta_r^j\right)_t (a_t' y^r - b_t) + \sum_{t\in T} \left(\beta_r^j\right)_t (b_t - a_t' x^r) \right) \\
&\leq \left(\liminf_j \sum_{t\in T} \left(\beta_r^j\right)_t \right) \mu_r \|a_t\| \\
&\leq \alpha^{-1} L \left\|x^r - y^r\right\| \mu_r \|a_t\| \\
&\leq \alpha^{-1} L N \left\|x^r - y^r\right\| \mu_r,
\end{aligned}$$

then

$$\frac{\alpha}{LN} \left\|x^r - y^r\right\| \leq \mu_r,$$

on the other hand

$$\frac{\alpha}{(L+1)(N+1+\delta)} < \frac{\alpha}{LN}.$$

Letting $\gamma = \frac{\alpha}{(L+1)(N+1+\alpha)}$ we get $\gamma \in (0,1)$ such that

$$\gamma \left\| x^r - y^r \right\| \leq \mu_r, \quad \text{for all } r = 0, 1, \ldots ,$$

when $0_n \in \operatorname{int} F$. If not, suppose that z is in the interior of F. Define $F_z := \{x \in \mathbb{R}^n : a_t' x \geq b_t - a_t' z, t \in T\}$, and $\mu_r(x,z) := \sup\left\{ \frac{b_t - a_t' x - a_t' z}{\|a_t\|} : t \in T_r \right\}$. As the origin is in the interior of F_z, we can prove that $\mu_r(x^r - z, z) \geq \gamma d(x^r - z, F_z)$ for all $r = 0, 1, 2, \ldots$. The above lemma follows from the fact $\mu_r = \mu_r(x^r - z, z)$ and $d(x^r, F) = d(x^r - z, F_z)$. ■

We would like to mention that the constants δ, N, B and L depend only on the nominal data and the initial point of the algorithm.

Theorem 7 *Let $\lambda_r \in [\nu, \mu]$, with $(0,2) \ni \nu < \mu \in (0,2)$ for all $r = 0, 1, 2, \ldots$. If $\operatorname{int} F \neq \emptyset$, $N < \infty$ and $B > 0$, then there exist $M > 2$, $0 < \theta < 1$ and $\bar{x} \in F$ such that $\bar{x} = \lim_r x^r$ and $\|x^r - \bar{x}\| \leq \theta^r \|x^0 - \bar{x}\|$ for all r big enough.*

Proof. From the definition of ε_r, we have $\varepsilon_r = \|x^r - x^{\acute{r}}\|$, where $x^{\acute{r}}$ is the orthogonal projection of x^r on the hyperplane H_{t_r}. We know that for every $r = 0, 1, \ldots,$

$$\varepsilon_r > \frac{\mu_r}{M}. \tag{15}$$

If we replace x by x^r, y by y^r and λ by λ_r in the inequality (6), from Lemma 5, Lemma 6 and the fact $\|x^{r+1} - y^{r+1}\|^2 \leq \|x^{r+1} - y^r\|^2$, we get

$$\begin{aligned}
\|x^{r+1} - y^{r+1}\|^2 &\leq \|x^{r+1} - y^r\|^2 \leq \|x^r - y^r\|^2 - \lambda_r (2 - \lambda_r) \|x^r - x^{\acute{r}}\|^2 \\
&= \|x^r - y^r\|^2 - \lambda_r (2 - \lambda_r) \varepsilon_r^2 \\
&\leq \|x^r - y^r\|^2 - \frac{\lambda_r (2 - \lambda_r) \mu_r^2}{M^2} \\
&\leq \|x^r - y^r\|^2 - \frac{\lambda_r (2 - \lambda_r) \gamma^2}{M^2} \|x^r - y^r\|^2 \\
&= \|x^r - y^r\|^2 (1 - \lambda_r (2 - \lambda_r) \gamma^2 M^{-2}).
\end{aligned} \tag{16}$$

$1 \geq \lambda_r (2 - \lambda_r) \geq \min\left[\nu (2 - \nu), \mu (2 - \mu)\right] = \zeta > 0$, for each $r = 0, 1, \ldots$. Letting $0 < \sigma = (1 - \zeta \gamma^2 M^{-2})^{\frac{1}{2}} < 1$ (choosing M large enough) and taking into account (16) repeatedly, we have

$$\|x^{r+1} - y^{r+1}\| \leq \sigma^{r+1} \|x^0 - y^0\|.$$

Since $\bar{x}$ and x^r are in the ball $B_{\|x^r-y^r\|}(y^r)$ for each $r = 0, 1, 2, ...$, we obtain

$$\frac{1}{2}\|x^{r+1} - \bar{x}\| \le \|x^{r+1} - y^{r+1}\| \le \sigma^{r+1}\|x^0 - y^0\| \le \sigma^{r+1}\|x^0 - \bar{x}\|,$$

which proves the theorem for any $\sigma < \theta < 1$. ■

4 Experimental test

The current investigation involves a variation of the parameter λ_r for each $r = 0, 1, \ldots$. We consider three examples. The first one, Example 8, is from previous experiments when λ_r has been considered fixed, see [8]. The next test problem, Example 9, we have proposed, is an usual problem appearing in the study of linear inequality systems. In these examples, for all $x, y \in F$ and for the initial guess, x^0, we have $\|x - y\| \lll d(F, x^0)$. With this, we try to illustrate what happens if the initial point is not that close to the feasible set. The last test problem, Example 10, distinguishes from the first two ones, because the index set, in the description of the feasible set, involves two indexes. It is a particular case of a system which appears in [4].

Algorithm 1 was coded in MATLAB. For the first two examples, we use the `fminbnd` script for minimizing the slack function (Step 2), whereas in the last example we use the `fmincon`,script dealing with optimization problems of multivariate functions. We need this due to the fact that the relaxation method uses these routines for solving the global optimization problem appearing in Step 2. In the three examples, the routine finishes when $\inf_{t\in T} g(x^r, t) \ge \epsilon$, considering ϵ, as a tolerance, that we establish as $\epsilon = 1 \times 10^{-8}$.

The initial guess, x^0, in Example 8 was taken as it was reported in [8], whereas in the remaining two examples it was randomly generated. Also, we establish the value of the parameters, β and M, as $\beta = 1 \times 10^{-4}$, and $M = 1000$.

In order to get an idea of the performance of Algorithm 1, with respect to previous ones, we classify the results in each example in two different cases. The first one, is devoted to the case when the parameter of relaxation λ remains constant within the interval $(0, 2]$, and the other case is when this parameter is chosen randomly in each iteration within the same range, under the conditions given in Theorem 3.

Tables 6, 8 and 11, concentrate the results corresponding to the analysis of fixed values of λ in the implementation. The first column presents the

λ value considered, while the second, denoted by `iter`, shows the number of iterations used by the implementation. The third column presents the last approximation, where the condition $\inf_{t\in T} g(x^r,t) \geq \epsilon$ has been reached. Finally, the fourth column indicates if the last point is either an exact feasible solution, i.e. when $\inf_{t\in T} g(x^r,t) > 0$, or it is an $\epsilon-$feasible solution, i.e. when $\inf_{t\in T} g(x^r,t) \geq \epsilon$ has been attained.

Tables 7-10, 2-5 and 12-15 show the results for different values of the parameter λ.

All experiments were carried on a Workstation of Intel(R) Xeon(R) CPU E5-2667 0 @ 2.90 GHz 2.90 GHz processor, 12 GB of RAM and Windows 7 operating system. Figures 1-2 show the feasible set (region in blank), for the first two examples.

4.1 Tests with fixed and random values of λ_r

Example 8 *The next problem under our consideration is*

$$\sigma = \left\{2tx_1 - \left(3t^2-3\right)x_2 \geq -2\left(t^4+3\right) : t\in[-\pi,\pi]\right\}.$$

With the initial guess $x^0 = (34.368772, 82.066698)'$, *Tables 1-8 show the efficiency of the methods and illustrate the effect of* λ, *we mentioned that in all cases the stopping criteria* $\inf g_{t\in T}(x^r,t)$ *was positively satisfied, i.e. we had exact solutions.*

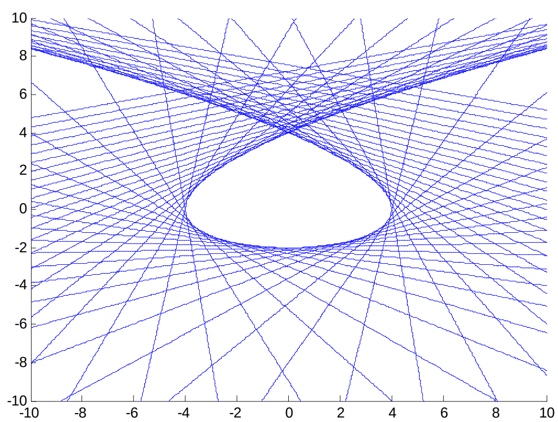

Figure 1: The feasible set F of the Example 8.

λ	`iter`	$\hat{x}$
0.1	339	$(0.391963, -0.044444)'$
0.4	74	$(0.391925, -0.044544)'$
0.7	35	$(0.391315, -0.046121)'$
1.0	15	$(0.399966, -0.002977)'$
1.2	8	$(-0.366915, 0.097708)'$
1.5	10	$(-0.346993, -0.027652)'$
1.8	16	$(-0.062166, 0.308286)'$
2.0	97	$(0.069442, -0.186739)'$

Table 1: Reports for Example 8 where fixed values of λ_r are considered.

<table>
<tr><th>λ</th><th>ν</th><th><code>iter</code></th><th>$\hat{x}$</th></tr>
<tr><td>0.0100</td><td rowspan="7">0.01</td><td rowspan="7">7</td><td rowspan="7">$(-3.986189510402, 0.130455699566)'$</td></tr>
<tr><td>0.401225</td></tr>
<tr><td>0.509657</td></tr>
<tr><td>1.235929</td></tr>
<tr><td>0.951845</td></tr>
<tr><td>0.709802</td></tr>
<tr><td>1.663349</td></tr>
</table>

Table 2: Reports for Example 8 with $\nu = 0.01$.

<table>
<tr><th>λ</th><th>ν</th><th><code>iter</code></th><th>$\hat{x}$</th></tr>
<tr><td>0.5000</td><td rowspan="3">0.5</td><td rowspan="3">6</td><td rowspan="3">$(-3.451416941731, -0.199954327799)'$</td></tr>
<tr><td>1.893895</td></tr>
<tr><td>1.024976</td></tr>
</table>

Table 3: Reports for Example 8 with $\nu = 0.5$.

<table>
<tr><th>λ</th><th>ν</th><th><code>iter</code></th><th>$\hat{x}$</th></tr>
<tr><td>1.000000</td><td rowspan="2">1</td><td rowspan="2">2</td><td rowspan="2">$(-1.087986291188, 2.495442730177)'$</td></tr>
<tr><td>1.243525</td></tr>
</table>

Table 4: Reports for Example 8 with $\nu = 1.0$.

λ	ν	`iter`	$\hat{x}$
1.500000	1.5	6	$(-0.419092799267, 3.376048821209)'$
1.628754			
1.920359			
1.627141			
1.907142			

Table 5: Reports for Example 8 with $\nu = 1.5$.

Example 9 *In this example, the test problem is in $\mathbb{R}^2$ correspond to the next system:*

$$\sigma = \left\{ \left(20 - 30t^2\right) x_1 + (60t + 20)x_2 \geq -4t^3 - 3t^4 + 18t^2 - 16 : t \in [-1.1, 1.7] \right\}.$$

We show the efficiency of the methods and illustrate the effect of λ. The initial guess is $x^0 = (53.610032, -33.575231)'$.

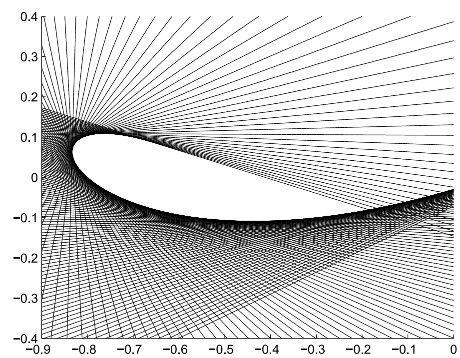

Figure 2: The feasible set F of the Example 9.

Results about the above example are summarized in Tables 6-10. Table 6, show that for λ_r values corresponding to 0.1 and 0.5 when the stopping criteria was satisfied we had $\epsilon-$solutions, note the negative value of the marginal function, whereas in Tables 7-10 the method always finished with an exact feasible solution, because $\inf g_{t\in T}(x^r, t) > 0$ as establish in Step 2 of the Algorithm 1.

λ	`iter`	$\hat{x}$	`gt`
0.1	480	$(0.81505979, 0.383076148)'$	-9.15×10^{-09}
0.5	68	$(0.639255031, 0.282980495)'$	-8.46×10^{-09}
1.0	10	$(0.542531751, 0.22791043)'$	4.44×10^{-16}
1.5	20	$(-0.823514404, 0.066059073)'$	0.0156
2.0	592	$(-0.803878176, 0.077015633)'$	0.0369

Table 6: Reports for Example 9 considering fixed values of λ_r.

λ	ν	`iter`	$\hat{x}$
0.1			
1.429787			
1.313208			
0.163847			
0.230732			
0.707239			
1.108642			
1.343447			
0.874476	0.1	17	$(-0.819929361, 0.076496562)'$
1.657964			
1.464882			
1.940434			
1.109534			
0.717777			
0.300695			
1.260821			

Table 7: Reports for Example 9 starting with $\nu = 0.1$.

λ	ν	`iter`	$\hat{x}$
0.5	0.5	10	$(-0.787758587893, -0.010860426097)'$
1.668203			
1.135179			
0.636235			
0.899707			
0.730485			
0.921508			
1.160128			
1.290714			

Table 8: Reports for Example 9 starting with $\nu = 0.5$.

λ	ν	`iter`	$\hat{x}$
1	1.0	10	$(-0.644373579970, -0.056080101579)'$
1.668203			
1.135179			
0.636235			
0.899707			
0.730485			
0.921508			
1.160128			
1.290714			

Table 9: Reports for Example 9 starting with $\nu = 1.0$.

λ		ν	iter	$\hat{x}$
1.5	1.731225	1.5	25	$(-0.151268457014, -0.049619232160)'$
1.61202	1.712175			
1.833916	1.730458			
1.922196	1.88508			
1.672231	1.661236			
1.89026	1.89237			
1.837666	1.735679			
1.503358	1.517881			
1.801085	1.587937			
1.693386	1.860879			
1.957996	1.736743			
1.500576	1.576361			

Table 10: Reports for Example 9 starting with $\nu = 1.5$.

Example 10 *Consider the system with $n = 3$ and T given by $T = \{t \in \mathbb{R}^2 \,|t_1 \in (1, \infty), t_2 \in [-3, 3]\}$ that is a particular case of the system given in [3].*

$$\sigma = \{-(t_1 + t_2 + 1)\, x_1 - 2t_2 x_2 + (t_1 - 2)\, x_3 \geq -t_1 + 2t_2 + 1, t \in T\}.$$

With random initial guess $x^0 = (53.610032, -33.575231, 234)'$ we have the following results summarize in Tables 11-15. We mentioned that in all cases we obtained exact solution in the sense of the previous example.

λ	iter	$\hat{x}$
0.1	802	$(-76.59096902, 37.29501823, 151.1819372)'$
0.5	197	$(-76.59094941, 37.29521194, 151.1818986)'$
1.0	66	$(-76.59092466, 37.29545837, 151.1818493)'$
1.5	7	$(-92.75218464, 43.40274114, 141.5743986)'$
2.0	8	$(-154.2997887, 79.41371155, 171.7701319)'$

Table 11: Reports for Example 10 where fixed values of λ_r are considered.

λ		ν	iter	$\hat{x}$
0.1	0.476425	0.1	20	$(-80.4578265, 37.6500462, 149.1597515)'$
1.348034	1.030407			
1.293149	0.745037			
0.65477	1.908098			
0.920137	1.848631			
0.129426	0.200086			
1.969721	1.50193			
0.41762	0.611327			
0.301811	0.903388			
0.807579	1.140955			

Table 12: Reports for Example 10 with initial $\nu = 0.1$.

λ	ν	iter	$\hat{x}$
0.5	0.5	7	$(-78.21645032, 37.45148494, 150.3300799)'$
1.086407			
1.74707			
1.705047			
0.590707			
1.098887			
1.290314			

Table 13: Reports for Example 10 with initial $\nu = 0.5$.

λ	ν	iter	$\hat{x}$
1	1	4	$(-102.8295169, 61.99435816, 131.8878283)'$
1.942737			
1.417744			
1.983052			
1.301455			

Table 14: Reports for Example 10 with initial $\nu = 1.0$.

λ	ν	ite	$\hat{x}$
1.5			
1.850549			
1.833169			
1.769563	1.5	6	$(-125.456067, 66.38939904, 137.215609)'$
1.849053			
1.833264			
1.589066			

Table 15: Reports for Example 10 with initial $\nu = 1.5$.

5 Conclusions

Prior work has documented the effectiveness of the extended relaxation method for solving LSIS, for example, [9], reports that over projections, for all iterations using a value fixed relaxation with $\lambda \in [1, 2]$, improve outcomes by decreasing the number of iterations to reach a feasible point. However, these studies have not considered the overall study where relaxation parameter variations during the process is considered.

Implemented results were compared with actual examples where were considered these variations as Tables 8-11 reported.

As tables shown, when a random election of λ is done, the number of iteration were significantly lower than for fixed values. What illustrating that in practice is preferable work with a random choice of the parameter λ.

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